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Complementary Variational Principles for Steady Heat Conduction with Mixed Boundary Conditions

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SUMMARY

New stationary, maximum and minimum principles associated with the boundary value problem of steady heat conduction with general boundary conditions are derived in a unified manner from the theory of complementary variational principles. One of the results contains the Brand–Lahey [3] stationary principle as a special case.

1. Introduction

There has been considerable interest recently in formulating the problem of heat conduction as a variational problem [1, 2, 3]. For example, Hays [1] has given an integral which is stationary for the temperature distribution satisfying the heat conduction equation in a region R, provided that at all points of the surface of R either the temperature is prescribed or the normal heat flux vanishes. Butler [2] gives a much simpler integral for steady problems with the same type of boundary conditions.

However, one might have the normal heat flux prescribed, rather than vanishing, on a portion of the bounding surface, or the even more complicated case when neither the temperature nor the normal heat flux is given but rather a relation between them as, for example, for a surface heated or cooled by convection or radiation. Stationary principles associated with these more complicated problems have been discussed by Brand and Lahey [3]. In this paper we derive new maximum and minimum principles for such problems from the theory of complementary variational principles [4]. One of our results contains the stationary principle of Brand and Lahey [3] as a special case.

2. Problem I

We consider the problem of finding the steady temperature distribution τ in a region R bounded by a surface S, subject to the boundary conditions that (i) τ is prescribed on S_1, a portion of S, and (ii) the normal heat flux $\boldsymbol{n}\cdot K(\tau)\,\mathrm{grad}\,\tau$ is prescribed on $S_2 = S - S_1$, where $K(\tau)$, the thermal conductivity, is some known function of τ. The equations of this problem are therefore

$$\mathrm{div}\,[K(\tau)\,\mathrm{grad}\,\tau] = 0 \quad \text{in } R, \tag{2.1}$$

with

$$\tau = \tau_B \quad \text{on } S_1\,, \tag{2.2}$$

$$\boldsymbol{n}\cdot K(\tau)\,\mathrm{grad}\,\tau = q_B \quad \text{on } S_2\,, \tag{2.3}$$

where τ_B and q_B are known functions on the boundary.

We now transform to a new dependent variable φ by setting

$$\varphi = \int_{\tau_0}^{\tau} K(v)\,dv\,, \tag{2.4}$$

where τ_0 is an arbitrary constant. It follows from (2.4) that

$$\mathrm{grad}\,\varphi = K(\tau)\,\mathrm{grad}\,\tau\,. \tag{2.5}$$

Also we define

$$\varphi_B = \int_{\tau_0}^{\tau_B} K(v)\, dv\,, \tag{2.6}$$

which is the prescribed value of φ on S_1 corresponding to (2.2). Hence the problem described by equations (2.1) – (2.3) becomes

$$\nabla^2 \varphi = 0 \text{ in } R\,, \tag{2.7}$$

with

$$\varphi = \varphi_B \text{ on } S_1\,, \tag{2.8}$$

$$\boldsymbol{n}\cdot \text{grad}\, \varphi = q_B \text{ on } S_2\,. \tag{2.9}$$

The variational problem associated with this boundary value problem will now be discussed.

3. Variational Principles for Problem I

To use the theory of complementary variational principles [4] we rewrite (2.7) in the canonical form

$$\text{grad}\, \Phi = \boldsymbol{U} = \frac{\partial H}{\partial \boldsymbol{U}} \text{ in } R\,, \tag{3.1}$$

$$-\text{div}\, \boldsymbol{U} = 0 = \frac{\partial H}{\partial \Phi} \text{ in } R\,, \tag{3.2}$$

with

$$\Phi = \varphi_B \text{ on } S_1\,, \tag{3.3}$$

and

$$\boldsymbol{n}\cdot \boldsymbol{U} = q_B \text{ on } S_2\,. \tag{3.4}$$

A suitable Hamiltonian H in (3.1) and (3.2) is given by

$$H(\boldsymbol{U}, \Phi) = \tfrac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U}\,. \tag{3.5}$$

The exact solution of this problem will be denoted by $(\boldsymbol{u}, \varphi)$. Now we introduce the associated generalized action functional [4, page 21].

$$I(\boldsymbol{U}, \Phi) = \int_R [\boldsymbol{U}\cdot \text{grad}\, \Phi - H(\boldsymbol{U}, \Phi)]\, dV + [\text{boundary terms}]\,. \tag{3.6}$$

In the present case this functional takes the form

$$I(\boldsymbol{U}, \Phi) = \int_R \boldsymbol{U}\cdot \text{grad}\, \Phi\, dV - \tfrac{1}{2}\int_R \boldsymbol{U}\cdot\boldsymbol{U}\, dV - \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{U}(\Phi - \varphi_B)\, dS - \int_{S_2} q_B \Phi\, dS\,, \tag{3.7}$$

$$= \int_R (-\text{div}\, \boldsymbol{U})\Phi\, dV - \tfrac{1}{2}\int_R \boldsymbol{U}\cdot\boldsymbol{U}\, dV + \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{U}\varphi_B\, dS + \int_{S_2} (\boldsymbol{n}\cdot\boldsymbol{U} - q_B)\Phi\, dS\,. \tag{3.8}$$

The surface integrals in (3.7) and (3.8), corresponding to the mixed boundary conditions (3.3) and (3.4), were derived by Arthurs [5].

The following results are readily verified.

3(a). *First variational principle.* For arbitrary independent functions $\boldsymbol{U}$, Φ the functional $I(\boldsymbol{U}, \Phi)$ is stationary at $(\boldsymbol{u}, \varphi)$, the solution pair of the boundary value problem given by (3.1) – (3.4).

3(b). *Second variational principle.* Let Φ be an admissible function, which need not satisfy any boundary conditions. Then using (3.7) we define a functional $J(\Phi)$ by

$$J(\Phi) = I(\text{grad}\ \Phi, \Phi) \tag{3.9}$$

$$= \tfrac{1}{2} \int_R (\text{grad}\ \Phi)^2 dV - \int_{S_1} (\Phi - \varphi_B)\, \boldsymbol{n} \cdot \text{grad}\ \Phi\, dS - \int_{S_2} q_B \Phi\, dS \tag{3.10}$$

$$= I(\boldsymbol{u}, \varphi) + \delta^2 J\,, \tag{3.11}$$

where

$$\delta^2 J = \tfrac{1}{2} \int_R [\text{grad}\,(\Phi - \varphi)]^2 dV - \int_{S_1} (\Phi - \varphi_B)\, \boldsymbol{n} \cdot \text{grad}\,(\Phi - \varphi)\, dS \tag{3.12}$$

is the second variation. Thus $J(\Phi)$ in (3.10) is stationary at φ.

3(c). *Third variational principle.* Let $\boldsymbol{U}$ be an admissible function which satisfies the conditions

$$\text{div}\ \boldsymbol{U} = 0 \ \text{ in } \ V\,, \tag{3.13}$$

$$\boldsymbol{n} \cdot \boldsymbol{U} = q_B \ \text{ on } \ S_2\,. \tag{3.14}$$

Using (3.8) we define the functional $G(\boldsymbol{U})$ by

$$G(\boldsymbol{U}) = I(\boldsymbol{U}, \Phi)\,,\ [\boldsymbol{U} \text{ subject to conditions (3.13) and (3.14)}] \tag{3.15}$$

$$= -\tfrac{1}{2} \int_R \boldsymbol{U} \cdot \boldsymbol{U}\, dV + \int_{S_1} \boldsymbol{n} \cdot \boldsymbol{U} \varphi_B\, dS \tag{3.16}$$

$$= I(\boldsymbol{u}, \varphi) + \delta^2 G\,, \tag{3.17}$$

where

$$\delta^2 G = -\tfrac{1}{2} \int_R (\boldsymbol{U} - \boldsymbol{u})^2 dV \tag{3.18}$$

is the second variation. Thus $G(\boldsymbol{U})$ in (3.16) is stationary at $\boldsymbol{u}$.

Since the exact function $\boldsymbol{u}$ is related to φ by $\boldsymbol{u} = \text{grad}\ \varphi$, it is desirable to choose the function $\boldsymbol{U}$ to have the form

$$\boldsymbol{U} = \text{grad}\ \Psi\,, \tag{3.19}$$

where Ψ is intended to be an approximation to φ. Then from (3.16)

$$G(\text{grad}\ \Psi) = -\tfrac{1}{2} \int_R (\text{grad}\ \Psi)^2 dV + \int_{S_1} \varphi_B \frac{\partial \Psi}{\partial n}\, dS\,, \tag{3.20}$$

where by (3.13) and (3.14) the function Ψ must satisfy the essential conditions

$$\nabla^2 \Psi = 0 \ \text{ in } \ V\,, \tag{3.21}$$

$$\frac{\partial \Psi}{\partial n} = q_B \ \text{ on } \ S_2\,. \tag{3.22}$$

3(d). *Minimum principle.* If the function Φ in (3.10) is made to satisfy the boundary condition

$$\Phi = \varphi_B \ \text{ on } \ S_1\,, \tag{3.23}$$

we see from (3.12) that $\delta^2 J$ is non-negative. Hence, by (3.11), we obtain the minimum principle

$$I(\boldsymbol{u}, \varphi) \leqq J(\Phi)\,, \tag{3.24}$$

where

$$J(\Phi) = \tfrac{1}{2} \int_R (\text{grad}\ \Phi)^2 dV - \int_{S_2} q_B \Phi\, dS \qquad (\Phi = \varphi_B \ \text{ on } \ S_1)\,. \tag{3.25}$$

3(e). *Maximum principle.* Since $\delta^2 G$ in (3.18) is non-positive, it follows from (3.17) that the maximum principle

$$G(\text{grad } \Psi) \leqq I(\boldsymbol{u}, \varphi) \tag{3.26}$$

holds, where

$$G(\text{grad } \Psi) = -\tfrac{1}{2}\int_R (\text{grad } \Psi)^2 dV + \int_{S_1} \varphi_B \frac{\partial \Psi}{\partial n} dS \quad \left(\nabla^2 \Psi = 0 \text{ in V}, \frac{\partial \Psi}{\partial n} = q_B \text{on} S_2\right). \tag{3.27}$$

Having established our stationary and extremum principles, we now wish to consider the stationary principle of Brand and Lahey [3]. These authors introduce the functional [3, eqn. 12].

$$J_{BL} = \int_R \boldsymbol{q}\cdot\boldsymbol{q}\, dV + \int_{S_2} \boldsymbol{n}\cdot\boldsymbol{q} H\, dS, \tag{3.28}$$

where

$$\boldsymbol{q} = -\text{ grad } \Phi = -K(T) \text{ grad } T, \tag{3.29}$$

$$H = \Phi = \int_{\tau_0}^{T} K(v) dv. \tag{3.30}$$

Here Φ is our trial function which is related to the temperature trial function T by (3.30), giving $\Phi = \phi$ when $T = \tau$ (see equations (2.4) and (2.5)). Hence (3.28) is (in our notation)

$$J_{BL} = \int_R (\text{grad } \Phi)^2 dV - \int_{S_2} (\boldsymbol{n}\cdot\text{grad } \Phi)\Phi\, dS. \tag{3.31}$$

Brand and Lahey [3] state that this functional is stationary for $T = \tau$ (i.e. $\Phi = \varphi$), for admissible functions which satisfy the given boundary conditions (2.2) and (2.3), viz.

$$\Phi = \varphi_B \text{ on } S_1, \tag{3.32}$$

and

$$\boldsymbol{n}\cdot\text{grad } \Phi = q_B \text{ on } S_2. \tag{3.33}$$

This statement is correct if the volume integral in (3.31) is multiplied by the factor $\frac{1}{2}$. Then we have the corrected Brand–Lahey functional

$$(\text{corrected})\ J_{BL} = \tfrac{1}{2}\int_R (\text{grad } \Phi)^2 dV - \int_{S_2} (\boldsymbol{n}\cdot\text{grad } \Phi)\Phi\, dS. \tag{3.34}$$

If conditions (3.32) and (3.33) are imposed on Φ then our functional $J(\Phi)$ in (3.10), reduces to (3.34). However, one of the main points of our stationary result for J in (3.10) is that *no* essential conditions whatever need be imposed on the trial function Φ. Extremum principles, as opposed to stationary principles, are another matter however, and we have seen in (3.24) that the *minimum* principle for $J(\Phi)$ holds only for functions Φ which do satisfy one of the boundary conditions, namely $\Phi = \varphi_B$ on S_1.

4. Problem II

This is the same as Problem I except that boundary condition (2.3) is replaced by

$$\boldsymbol{n}\cdot K(\tau) \text{ grad } \tau = f(\tau) \text{ on } S_2, \tag{4.1}$$

with f a specified function but neither $K(\tau)$ grad τ nor τ given on S_2. Using (2.4) – (2.6) we can reformulate Problem II as

$$\nabla^2 \varphi = 0 \text{ in } R, \tag{4.2}$$

$$\varphi = \varphi_B \text{ on } S_1, \tag{4.3}$$

$$\boldsymbol{n}\cdot\text{grad } \varphi = b(\varphi) \text{ on } S_2, \tag{4.4}$$

where

$$b(\varphi)=f(\tau), \tag{4.5}$$

φ and τ being related through (2.4). We now proceed to find variational principles for this boundary value problem.

5. Variational Principles for Problem II

As for Problem I we rewrite equation (4.2) in canonical form

$$\text{grad}\,\Phi = \boldsymbol{U} = \frac{\partial H}{\partial \boldsymbol{U}} \text{ in } R, \tag{5.1}$$

$$-\text{div}\,\boldsymbol{U} = 0 = \frac{\partial H}{\partial \Phi} \text{ in } R, \tag{5.2}$$

with

$$\Phi=\varphi_B \text{ on } S_1, \tag{5.3}$$

and

$$\boldsymbol{n}\cdot\boldsymbol{U}=b(\Phi) \text{ on } S_2. \tag{5.4}$$

The exact solution of this problem will be denoted by $(\boldsymbol{u}, \varphi)$. The associated generalized action functional is

$$I(\boldsymbol{U}, \Phi) = \int_R \boldsymbol{U}\cdot\text{grad}\,\Phi dV - \tfrac{1}{2}\int_R \boldsymbol{U}\cdot\boldsymbol{U} dV - \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{U}(\Phi-\varphi_B)dS - \int_{S_2} B(\Phi)dS, \tag{5.5}$$

$$= \int_R (-\text{div}\,\boldsymbol{U})\Phi dV - \tfrac{1}{2}\int_R \boldsymbol{U}\cdot\boldsymbol{U} dV + \int_{S_1} \boldsymbol{n}\cdot\boldsymbol{U}\varphi_B dS + \int_{S_2} [\boldsymbol{n}\cdot\boldsymbol{U}\Phi - B(\Phi)]dS, \tag{5.6}$$

where

$$B(\Phi) = \int_{\Phi_0}^{\Phi} b(v)dv. \tag{5.7}$$

This action functional differs from the functional $I(\boldsymbol{U}, \Phi)$ in (3.7) and (3.8) only in the last surface integral. The modification used here follows from the work of Arthurs [6] on boundary conditions of the form (5.4).

The following results are readily obtained.

5(*a*). *First variational principle.* For arbitrary independent functions $\boldsymbol{U}, \Phi$ the functional $I(\boldsymbol{U}, \Phi)$ in (5.5) and (5.6) is stationary at $(\boldsymbol{u}, \varphi)$, the solution pair of the boundary value problem described by equations (5.1)–(5.4).

5(*b*). *Second variational principle.* Let Φ be an admissible function, which need not satisfy any boundary conditions. Then using (5.5) we define a functional $J(\Phi)$ by

$$J(\Phi) = I(\text{grad}\,\Phi, \Phi) \tag{5.8}$$

$$= \tfrac{1}{2}\int_R (\text{grad}\,\Phi)^2 dV - \int_{S_1} (\Phi-\varphi_B)\boldsymbol{n}\cdot\text{grad}\,\Phi dS - \int_{S_2} B(\Phi)dS \tag{5.9}$$

$$= I(\boldsymbol{u}, \varphi) + \delta^2 J, \tag{5.10}$$

where

$$\delta^2 J = \tfrac{1}{2}\int_R [\text{grad}(\Phi-\varphi)]^2 dV - \int_{S_1} (\Phi-\varphi_B)\boldsymbol{n}\cdot\text{grad}(\Phi-\varphi)dS - \tfrac{1}{2}\int_{S_2} (\Phi-\varphi)^2 \overline{\frac{d^2 B}{d\Phi^2}} dS \tag{5.11}$$

is the second variation, the bar in the last integral indicating that the second derivative is evaluated for some function $\varphi+\varepsilon(\Phi-\varphi)$, $0\leq\varepsilon\leq 1$. Thus $J(\Phi)$ in (5.9) is stationary at φ.

5(*c*). *Third variational principle.* Let U be an admissible function which satisfies the essential condition

$$\operatorname{div} U = 0 \text{ in } V. \tag{5.12}$$

In (5.6) we take $\Phi = b^{-1}(n\cdot U)$ on S_2, assuming that the inverse b^{-1} exists. This defines a functional $G(U)$ by

$$G(U) = I(U, \Phi) \qquad (\Phi = b^{-1}(n\cdot U) \text{ on } S_2) \tag{5.13}$$

$$= -\tfrac{1}{2}\int_R U\cdot U\,dV + \int_{S_1} n\cdot U\varphi_B\,dS + \int_{S_2}\{n\cdot Ub^{-1}(n\cdot U) - B[b^{-1}(n\cdot U)]\}\,dS \tag{5.14}$$

$$= I(u, \varphi) + \delta^2 G, \tag{5.15}$$

where

$$\delta^2 G = -\tfrac{1}{2}\int_R (U-u)^2\,dV + \tfrac{1}{2}\int_{S_2} [b^{-1}(n\cdot U) - \varphi]^2\, \overline{\frac{d^2 B}{d\Phi^2}}\,dS \tag{5.16}$$

is the second variation. Thus $G(U)$ in (5.14) is stationary at u.

As in section 3, we take $U = \operatorname{grad}\Psi$, where Ψ is intended to be an approximation to φ. Then from (5.14)

$$G(\operatorname{grad}\Psi) = -\tfrac{1}{2}\int_R (\operatorname{grad}\Psi)^2\,dV + \int_{S_1}\varphi_B\frac{\partial\Psi}{\partial n}\,dS + \int_{S_2}\left\{\frac{\partial\Psi}{\partial n}b^{-1}\left(\frac{\partial\Psi}{\partial n}\right) - B\left[b^{-1}\left(\frac{\partial\Psi}{\partial n}\right)\right]\right\}dS, \tag{5.17}$$

where by (5.12) the function Ψ must satisfy the essential condition

$$\nabla^2\Psi = 0 \text{ in } V. \tag{5.18}$$

5(*d*). *Minimum principle.* If the function Φ in (5.9) satisfies

$$\Phi = \varphi_B \text{ on } S_1, \tag{5.19}$$

and if

$$\frac{d^2 B}{d\Phi^2} = \frac{db}{d\Phi} \leqq 0 \text{ for all } \Phi \text{ on } S_2, \tag{5.20}$$

it follows from (5.11) that $\delta^2 J$ is non-negative. Hence by (5.10) we obtain the minimum principle

$$I(u, \varphi) \leqq J(\Phi), \tag{5.21}$$

where

$$J(\Phi) = \tfrac{1}{2}\int_R (\operatorname{grad}\Phi)^2\,dV - \int_{S_2} B(\Phi)\,dS \tag{5.23}$$

subject to (5.19) and (5.20).

5(*e*). *Maximum principle.* If $B(\Phi)$ satisfies (5.20), it follows from (5.16) that $\delta^2 G$ is non-positive. Hence by (5.15) we obtain the maximum principle

$$G(\operatorname{grad}\Psi) \leqq I(u, \varphi), \tag{5.23}$$

where $G(\operatorname{grad}\Psi)$ is given in (5.17) with Ψ subject to the condition (5.18).

Now that we have established our stationary and extremum principles for Problem II, we turn to the stationary principle of Brand and Lahey [3]. These authors introduce the functional, again using their notation, [3, eqn. 20].

$$J_{BL} = \int_R \boldsymbol{q} \cdot \boldsymbol{q} \, dV + \int_{S_2} g \, dS \,, \tag{5.24}$$

where

$$\boldsymbol{q} = -\text{ grad } \Phi = -K(T) \text{ grad } T \,, \tag{5.25}$$

and

$$g = g(T) = -\int^T K(\mu) f(\mu) \, d\mu \,. \tag{5.26}$$

We use g instead of their G to avoid confusion with our functional $G(U)$. In (5.25) Φ is our trial function which is related to T by

$$\Phi = \int_{\tau_0}^T K(v) dv \,. \tag{5.27}$$

If we set

$$\lambda = \int_{\tau_0}^{\mu} K(v) dv \tag{5.28}$$

then $\lambda = \Phi$ when $\mu = T$, and $\lambda = \varphi$ when $\mu = \tau$, and by (2.4) and (4.5) it follows that

$$b(\lambda) = f(\mu) \,. \tag{5.29}$$

Now consider

$$\begin{aligned} B(\Phi) &= \int_{\Phi_0}^{\Phi} b(\lambda) d\lambda \,, && \text{by definition (5.7),} \\ &= \int_{\mu_0}^{T} f(\mu) \frac{d\lambda}{d\mu} d\mu \,, && \text{by (5.29),} \\ &= \int_{\mu_0}^{T} f(\mu) K(\mu) d\mu \,, && \text{by (5.28),} \\ &= -g(T) \,, && \text{by (5.26)} \end{aligned} \tag{5.30}$$

Hence the Brand–Lahey functional (5.24) is, in our notation,

$$J_{BL} = \int_R (\text{grad } \Phi)^2 dV - \int_{S_2} B(\Phi) dS \,. \tag{5.31}$$

Brand and Lahey [3] state that this functional is stationary for $T = \tau$ (i.e. $\Phi = \varphi$), for admissible functions which satisfy the given boundary condition (4.3), viz.,

$$\Phi = \varphi_B \text{ on } S_1 \,. \tag{5.32}$$

This is correct if the volume integral in (5.31) is multiplied by the factor $\frac{1}{2}$. Then we have the corrected functional

$$\text{(corrected) } J_{BL} = \tfrac{1}{2} \int_R (\text{grad } \Phi)^2 dV - \int_{S_2} B(\Phi) dS \,. \tag{5.33}$$

If condition (5.32) is imposed on Φ, then our functional $J(\Phi)$ in (5.9) reduces to (5.33). However, to obtain our stationary result for $J(\Phi)$ in (5.9), *no* essential condition need be imposed on the trial function Φ.

In the special case when $f(\tau)$ does not depend on the temperature but rather is a prescribed function q_B on the boundary S_2, these results for Problem II reduce, as they should, to those for Problem I.

REFERENCES

[1] D. F. Hays, Variational formulation of the heat equation, Chapter 2 in *Non-equilibrium thermodynamics, variational techniques, and stability*. R. J. Donnelly *et al.*, eds. U. of Chicago Press, (1966).
[2] H. W. Butler, *A minimum principle for heat conduction*, Proc. 5th. U.S. National Congress of Appl. Mech. A.S.M.E., New York(1966) 739.
[3] R. S. Brand and F. J. Lahey, Variational principles for steady heat conduction with mixed boundary conditions, *J. Engineering Math.*, 3 (1969) 119–121.
[4] A. M. Arthurs, *Complementary variational principles*, Clarendon Press, Oxford, (1970).
[5] A. M. Arthurs, A note on Komkov's class of boundary value problems and associated variational principles, *J. Math. Anal. Appl.*, 33 (1971) 402–407.
[6] A. M. Arthurs, Extremum principles for a class of boundary-value problems, *Proc. Camb. Phil. Soc.*, 65 (1969) 803–806.

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